

Pricing of basket options I

Alexander Kushpel
 Department of Mathematics,
 University of Leicester, LE1 7RH, UK
 E-mail: ak412@le.ac.uk

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Abstract

Pricing of high-dimensional options is a deep problem of the Theoretical Financial Mathematics. In this article we give a transparent and self contained treatment of this problem. Namely, we present and study a new class of Lévy driven models of stock markets. In our opinion, any market model should be based on a transparent and intuitively easily acceptable pre-axiomatic concept. In our case this is the system of stochastic equations (2). Our market model is based on the principle of inheritance, i.e. for the particular choice of parameters it coincides with known models. Also, our model is effectively numerically realisable. For the class of models proposed, we give an explicit representations of characteristic functions. This allows us to construct a sequence of approximation formulas to price basket options. We show that our approximation formulas have almost optimal rate of convergence in the sense of respective n -widths.

Keywords: approximation, Lévy driven models, Fourier transform, reconstruction.

Subject: 91G20, 60G51, 91G60, 91G80.

1 Introduction

Consider a frictionless market with no arbitrage opportunities and a constant riskless interest rate $r > 0$. Let $S_{j,t}$, $1 \leq j \leq n$, $t \geq 0$, be n asset price processes. Consider a European call option on the price spread $S_{1,T} - \sum_{j=1}^n S_{j,T}$. The common spread option with maturity $T > 0$ and strike $K \geq 0$ is the contract that pays $\left(S_{1,T} - \sum_{j=1}^n S_{j,T} - K\right)_+$ at time T , where $(a)_+ := \max\{a, 0\}$. There is a wide range of such options traded across different sectors of financial markets. For instance, the crack spread and crush spread options in the commodity markets [21], [26], credit spread options in the fixed income markets, index spread

options in the equity markets [7] and the spark (fuel/electricity) spread options in the energy markets [6], [24].

Assuming the existence of a risk-neutral equivalent martingale measure we get the following pricing formula for the value at time 0,

$$V = e^{-rT} \mathbb{E}[\varphi],$$

where φ is a reward function and the expectation is taken with respect to the equivalent martingale measure. Usually, the reward function has a simple structure. In particular, in the case of call option,

$$\varphi = \left(S_{1,T} - \sum_{j=1}^n S_{j,T} - K \right)_+$$

Hence the main problem is to approximate properly the respective density function and then to approximate $\mathbb{E}[\varphi]$. There is an extensive literature on spread options and their applications. In particular, if $K = 0$ a spread option is the same as an option to exchange one asset for another. An explicit solution in this case has been obtained by Margrabe [18]. Margrabe's model assumes that $S_{1,t}$ and $S_{2,t}$ follow a geometric Brownian motion whose volatilities σ_1 and σ_2 do not need to be constant, but the volatility σ of $S_{1,t}/S_{2,t}$ is a constant, $\sigma = (\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho)^{1/2}$, where ρ is the correlation coefficient of the Brownian motions $S_{1,t}$ and $S_{2,t}$. Margrabe's formula states that

$$V = e^{-q_1 T} S_{1,0} N(d_1) - e^{-q_2 T} S_{2,0} N(d_2),$$

where N denotes the cumulative distribution for a standard Normal distribution,

$$d_1 = \frac{1}{\sigma T^{1/2}} \left(\ln \left(\frac{S_{1,0}}{S_{2,0}} \right) + \left(q_1 - q_2 + \frac{\sigma}{2} \right) T \right)$$

and $d_2 = d_1 - \sigma T^{1/2}$.

Unfortunately, in the case where $K > 0$ and $S_{1,t}$, $S_{2,t}$ are geometric Brownian motions, no explicit pricing formula is known. In this case various approximation methods have been developed. There are three main approaches: Monte Carlo techniques which are most convenient for high-dimensional situation because the convergence is independent of the dimension, fast Fourier transform methods studied in [3] and PDEs. Observe that PDE based methods are suitable if the dimension of the PDE is low (see, e.g. [23], [8], [27] and [28] for more information). The usual PDE's approach is based on numerical approximation resulting in a large system of ordinary differential equations which can then be solved numerically.

Approximation formulas usually allow quick calculations. In particular, a popular among practitioners Kirk formula [14] gives a good approximation to the spread call (see also Carmona-Durrleman procedure [4], [15]). Various applications of fast Fourier transform have been considered in [5] and [16].

It is well-known that Merton-Black-Scholes theory becomes much more efficient if additional stochastic factors are introduced. Consequently, it is important to consider a wider family of Lévy processes. Stable Lévy processes have been used first in this context by Mandelbrot [17] and Fama [12].

From the 90th Lévy processes became very popular (see, e.g., [19], [20], [1], [2] and references therein).

2 High-dimensional Lévy driven models

In this section we introduce a class of stochastic systems to model multidimensional return processes.

Let $X_{1,t}, \dots, X_{n,t}$ and $Z_{1,t}, \dots, Z_{n,t}$ be independent random variables, with the densities functions $f_{1,t}(x_1), \dots, f_{n,t}(x_n)$ and $z_{1,t}(x_1), \dots, z_{n,t}(x_n)$ and characteristic exponents ψ_s and $\phi_m, 1 \leq s, m \leq n$ respectively. Let $\mathbf{X}_t = (X_{1,t}, \dots, X_{n,t})^T$, $\mathbf{Z}_t = (Z_{1,t}, \dots, Z_{n,t})^T$ and $\mathbf{A} = (a_{j,k})$ be a real matrix of size $n \times n$. Consider random vector $\mathbf{U}_t = (U_{1,t}, \dots, U_{n,t})^T$,

$$\mathbf{U}_t = \mathbf{X}_t + \mathbf{A}\mathbf{Z}_t. \quad (1)$$

A matrix \mathbf{A} reflects dependence between the return processes $U_{1,t}, \dots, U_{n,t}$ in our model. Assume for simplicity that $\mathbb{E}[X_{s,t}] = 0$ and $\mathbb{E}[Z_{s,t}] = 0, 1 \leq s \leq n$. It is easy to check that for any s and $l, 1 \leq s \neq l \leq n$ the correlation coefficient $\rho(U_{s,t}, U_{l,t})$ between $U_{s,t}$ and $U_{l,t}$, where

$$U_{s,t} = X_{s,t} + \sum_{k=1}^n a_{s,k} Z_{s,t}, U_{l,t} = X_{l,t} + \sum_{k=1}^n a_{l,k} Z_{l,t}$$

is

$$\begin{aligned} \rho(U_{s,t}, U_{l,t}) &= \frac{\mathbb{E}[U_{s,t}U_{l,t}]}{\sqrt{\mathbb{E}[U_{s,t}^2] \mathbb{E}[U_{l,t}^2]}} \\ &= \frac{\sum_{k=1}^n a_{s,k}^2 \text{var}(Z_{s,t})}{\left(\left(\text{var}(X_{s,t}) + \sum_{k=1}^n a_{s,k}^2 \text{var}(Z_{s,t}) \right) \left(\text{var}(X_{l,t}) + \sum_{k=1}^n a_{l,k}^2 \text{var}(Z_{l,t}) \right) \right)^{1/2}}. \end{aligned}$$

In particular, if $\text{var}(X_{s,t}) = \text{var}(Z_{s,t}) = v$ and $a_{s,k} = 1, 1 \leq s, k \leq n$ then $\rho(U_{s,t}, U_{l,t}) = n(n+1)^{-1}$. It reflects our empirical experience: if the market is in crisis then the prices of stocks are highly correlated.

The next statement gives us an explicit form of the characteristic function of the return process \mathbf{U}_t .

Theorem 1 *Let $\mathbf{U}_t = \mathbf{X}_t + \mathbf{A}\mathbf{Z}_t$, $\mathbf{A} = (a_{m,k})$ then in our notations the characteristic function $\Phi(\mathbf{v}, t)$ of \mathbf{U}_t has the form*

$$\Phi(\mathbf{v}, t) = (2\pi)^n \left(\prod_{s=1}^n \mathbf{F}^{-1}(f_{s,t}) \right) (v_s) \cdot \mathbf{F}^{-1} \left(\prod_{m=1}^n z_{m,t} \right) (\mathbf{A}^* \mathbf{v}),$$

$$= \prod_{s=1}^n \exp(-t\psi_s(v_s)) \cdot \prod_{m=1}^n \exp\left(-t\phi_m\left(\sum_{k=1}^n a_{k,m}v_k\right)\right),$$

where $A^* = (a_{k,m})$ is the conjugate of A .

Proof Consider transformation $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ defined as

$$\begin{aligned} \mathbf{U}_t &= \mathbf{X}_t + A\mathbf{Z}_t, \\ \mathbf{Z}_t &= \mathbf{Z}_t. \end{aligned} \tag{2}$$

Hence the inverse is given by

$$\begin{aligned} \mathbf{X}_t &= \mathbf{U}_t - A\mathbf{Z}_t, \\ \mathbf{Z}_t &= \mathbf{Z}_t. \end{aligned}$$

or

$$\begin{pmatrix} \mathbf{X}_t \\ \mathbf{Z}_t \end{pmatrix} = \begin{pmatrix} \mathbf{I} & -A \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{U}_t \\ \mathbf{Z}_t \end{pmatrix}$$

and the Jacobian J of this transform is

$$J = \det \begin{pmatrix} \mathbf{I} & -A \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = 1,$$

where $\mathbf{I} = \mathbf{I}_{n \times n}$ is an identity. The density function $\phi_t(\mathbf{u}_t, \mathbf{z}_t)$ is given by

$$\phi_t(\mathbf{u}, \mathbf{z}) = \prod_{s=1}^n f_{s,t} \left(u_s - \sum_{m=1}^n a_{s,m} z_m \right) \prod_{l=1}^n z_{l,t}(z_l).$$

It means that the density function $\omega_t(\mathbf{u})$ is

$$\omega_t(\mathbf{u}) = \int_{\mathbb{R}^n} \phi_t(\mathbf{u}, \mathbf{z}) d\mathbf{z}$$

and the characteristic function has the form

$$\begin{aligned} \Phi(\mathbf{v}, t) &:= \mathbb{E}[\exp(i\langle \mathbf{U}_t, \mathbf{v} \rangle)] := \exp(-t\psi(\mathbf{v})) = \mathbf{F}\omega_t(\mathbf{v}) \\ &= \int_{\mathbb{R}^n} \exp(i\langle \mathbf{u}, \mathbf{v} \rangle) \omega_t(\mathbf{u}) d\mathbf{u} \\ &= \int_{\mathbb{R}^n} \exp(i\langle \mathbf{u}, \mathbf{v} \rangle) \left(\int_{\mathbb{R}^n} \phi_t(\mathbf{u}, \mathbf{z}) d\mathbf{z} \right) d\mathbf{u} \\ &= \int_{\mathbb{R}^n} \exp(i\langle \mathbf{u}, \mathbf{v} \rangle) \left(\int_{\mathbb{R}^n} \prod_{s=1}^n f_{s,t} \left(u_s - \sum_{m=1}^n a_{s,m} z_m \right) \prod_{m=1}^n z_{m,t}(z_m) d\mathbf{z} \right) d\mathbf{u} \\ &= \int_{\mathbb{R}^n} \left(\prod_{s=1}^n \int_{\mathbb{R}} f_{s,t} \left(u_s - \sum_{m=1}^n a_{s,m} z_m \right) \exp(iu_s v_s) du_s \right) \prod_{m=1}^n z_{m,t}(z_m) d\mathbf{z}. \end{aligned} \tag{3}$$

Let $\xi_s = u_s - \sum_{m=1}^n a_{s,m} z_m$, $1 \leq s \leq n$ then

$$\begin{aligned}
& \int_{\mathbb{R}} f_{s,t} \left(u_s - \sum_{m=1}^n a_{s,m} z_m \right) \exp(i u_s v_s) du_s \\
&= \int_{\mathbb{R}} f_{s,t}(\xi_s) \exp \left(i \left(\xi_s + \sum_{m=1}^n a_{s,m} z_m \right) v_s \right) d\xi_s \\
&= \exp \left(i v_s \sum_{m=1}^n a_{s,m} z_m \right) \int_{\mathbb{R}} f_{s,t}(\xi_s) \exp(i \xi_s v_s) d\xi_s \\
&= \exp \left(i v_s \sum_{m=1}^n a_{s,m} z_m \right) 2\pi \mathbf{F}^{-1}(f_{s,t})(v_s)
\end{aligned} \tag{4}$$

Comparing 3 and 4 we get

$$\begin{aligned}
\Phi(\mathbf{v}, t) &= \int_{\mathbb{R}^n} \left(\prod_{s=1}^n \exp \left(i v_s \sum_{m=1}^n a_{s,m} z_m \right) 2\pi \mathbf{F}^{-1}(f_{s,t})(v_s) \right) \prod_{m=1}^n z_{m,t}(z_m) d\mathbf{z} \\
&= \prod_{s=1}^n 2\pi \mathbf{F}^{-1}(f_{s,t})(v_s) \int_{\mathbb{R}^n} \left(\prod_{s=1}^n \exp \left(i v_s \sum_{m=1}^n a_{s,m} z_m \right) \right) \prod_{m=1}^n z_{m,t}(z_m) d\mathbf{z} \\
&= \prod_{s=1}^n 2\pi \mathbf{F}^{-1}(f_{s,t})(v_s) \int_{\mathbb{R}^n} \exp \left(i \sum_{s=1}^n \left(v_s \sum_{m=1}^n a_{s,m} z_m \right) \right) \prod_{m=1}^n z_{m,t}(z_m) d\mathbf{z} \\
&= \prod_{s=1}^n 2\pi \mathbf{F}^{-1}(f_{s,t})(v_s) \int_{\mathbb{R}^n} \exp(\langle \mathbf{v}, \mathbf{A} \mathbf{z} \rangle) \left(\prod_{m=1}^n z_{m,t}(z_m) \right) d\mathbf{z} \\
&= \prod_{s=1}^n 2\pi \mathbf{F}^{-1}(f_{s,t})(v_s) \int_{\mathbb{R}^n} \exp(\langle \mathbf{A}^* \mathbf{v}, \mathbf{z} \rangle) \left(\prod_{m=1}^n z_{m,t}(z_m) \right) d\mathbf{z} \\
&= \prod_{s=1}^n 2\pi \mathbf{F}^{-1}(f_{s,t})(v_s) \cdot \mathbf{F}^{-1} \left(\prod_{m=1}^n 2\pi z_{m,t} \right) (\mathbf{A}^* \mathbf{v}) \\
&= \prod_{s=1}^n 2\pi \mathbf{F}^{-1}(f_{s,t})(v_s) \cdot \mathbf{F}^{-1} \left(\prod_{m=1}^n 2\pi z_{m,t} \right) (\mathbf{A}^* \mathbf{v}) \\
&= \prod_{s=1}^n 2\pi \mathbf{F}^{-1}(f_{s,t})(v_s) \cdot \mathbf{F}^{-1} \left(\prod_{m=1}^n 2\pi z_{m,t} \right) (\mathbf{A}^* \mathbf{v}),
\end{aligned}$$

where $\mathbf{A}^* = (a_{k,j})$ is a conjugate to \mathbf{A} . Hence

$$\Phi(\mathbf{v}, t) = \prod_{s=1}^n \exp(-t \psi_s(v_s)) \cdot \prod_{m=1}^n \exp \left(-t \phi_m \left(\sum_{k=1}^n a_{k,m} v_k \right) \right).$$

3 The equivalent martingale measure condition

In this section we specify an equivalent martingale measure condition for our model. Under the equivalent martingale measure all assets have the same expected rate of return which is a risk free rate. It simply means that under no-arbitrage conditions the risk preferences of investors acting on the market do not enter into valuation decisions. Consider a frictionless market consisting of a riskless bond B and stock S . In this market S is modeled by an exponential Lévy process $S = S_t = S_0 e^{X_t}$ under a chosen equivalent martingale measure \mathbb{Q} . Assume that the riskless rate r is constant. The next statement is a generalisation of a known result. In the previous versions authors assumed that the characteristic exponent ψ admits an analytic extension into the strip $\{z \mid -1 \leq \text{Im} z \leq 0\}$ (see e.g. [2]).

Theorem 2. *Let \mathbb{Q} be a chosen equivalent martingale measure and $D \subset \mathbb{R} + i\mathbb{R}$ be the domain of the characteristic exponent $\psi^{\mathbb{Q}}$. Assume that $\mathbb{R} \cup \{-i\} \subset D$, then in our notations $\psi^{\mathbb{Q}}(-i) = -r$.*

Proof The discount price process which is given by

$$\tilde{S}_t = \exp(-rt) S_t = \exp(-rt) S_0 \exp(X_t)$$

must be a martingale under a chosen equivalent martingale measure \mathbb{Q} , i.e. for any $0 \leq l < t \leq T$ the martingale condition must hold,

$$\tilde{S}_l = \mathbb{E}^{\mathbb{Q}} [\tilde{S}_t | \mathcal{F}_l].$$

In particular, let $l = 0$ then for any $t \in (0, T]$ we have

$$\begin{aligned} \tilde{S}_0 &= S_0 \exp(-r0) = S_0 = \mathbb{E}^{\mathbb{Q}} [S_0 \exp(-rt) \exp(X_t) | \mathcal{F}_0] \\ &= \mathbb{E}^{\mathbb{Q}} [S_0 \exp(-rt) \exp(X_t)] = S_0 \mathbb{E}^{\mathbb{Q}} [\exp(-rt) \exp(X_t)]. \end{aligned}$$

Since $S_0 > 0$ then $\mathbb{E}^{\mathbb{Q}} [\exp(-rt) \exp(X_t)] = 1$. Let $t = T$ then $\exp(rT) = \mathbb{E}^{\mathbb{Q}} [\exp(X_T)]$. Since $-i \in D$ then by the definition of the characteristic exponent

$$\exp(-T\psi^{\mathbb{Q}}(-i)) = \mathbb{E}^{\mathbb{Q}} [\exp(i(-i)X_T)] = \mathbb{E}^{\mathbb{Q}} [\exp(X_T)]. \quad (5)$$

Hence, since $T > 0$ then from (5) it follows that $r = -\psi^{\mathbb{Q}}(-i)$.

In general \mathbb{Q} is not unique. In what follows we assume that \mathbb{Q} has been fixed and all expectations will be computed with respect to this measure.

We specify now the equivalent martingale measure condition for the system (1).

Theorem 3. *Let the stock prices are modeled by*

$$S_{s,t} = S_{s,0} \exp(U_{s,t}), 1 \leq s \leq n.$$

and the domain $D \subset \mathbb{R}^n + i\mathbb{R}^n$ of the characteristic exponent $\psi^{\mathbb{Q}}$ contains $\mathbb{R}^n \cup (\cup_{k=1}^n \{-i\mathbf{e}_k\})$ where $\{\mathbf{e}_k, 1 \leq k \leq n\}$ is the standard basis in \mathbb{R}^n then

$$\psi^{\mathbb{Q}}(-i\mathbf{e}_s) = -r, 1 \leq s \leq n.$$

Proof Observe that for any $1 \leq s \leq n$ the discount price process $S_{s,t}$ must be a martingale under a chosen equivalent martingale measure \mathbb{Q} . Let $\psi_s^{\mathbb{Q}}(x_s)$ be the characteristic exponent of $U_{s,t}$ then

$$\begin{aligned}\exp(-t\psi_s^{\mathbb{Q}}(x_s)) &= \mathbb{E}^{\mathbb{Q}}[\exp(\langle ix_s, U_{s,t} \rangle)] \\ &= \mathbb{E}^{\mathbb{Q}}[\exp(\langle i\mathbf{x}, U_{s,t}\mathbf{e}_s \rangle)] = \exp(-t\psi_s^{\mathbb{Q}}(x_s\mathbf{e}_s))\end{aligned}$$

and by the Theorem 2 we get $r = -\psi_s^{\mathbb{Q}}(-i)$ which gives a system of n equations

$$\psi_s^{\mathbb{Q}}(-i\mathbf{e}_s) = -r, 1 \leq s \leq n.$$

Observe that in general riskless interest rate may depend on s . In this case we get the system $\psi_s^{\mathbb{Q}}(-i\mathbf{e}_s) = -r_s, 1 \leq s \leq n$.

4 KoBoL family

In this section we study characteristic exponents of KoBoL family. The idea is based on a simple observation. From the Lévi-Khintchine formula (8) it follows that it is possible to find $\psi(\xi)$ explicitly if we can find the inverse Fourier transform of $\Pi(dx)$. It was suggested by the authors of [2] to consider the following form of $\Pi(dx)$,

$$\Pi(dx) = |x|^\alpha \exp(-\beta|x|),$$

where α and β are fixed parameters.

A known class of high-dimensional models is based on so-called KoBoL family which is given by

$$\Pi(d\mathbf{x}) = \rho^{-\nu-1} \exp(-\lambda(\phi)\rho) d\rho \Pi'(d\phi),$$

where $\Pi'(d\phi)$ is a finite measure on the unit sphere \mathbb{S}^{n-1} and $\lambda : C(\mathbb{S}^{n-1}) \rightarrow \mathbb{R}_+$ [2]. The respective characteristic exponent has the form

$$\psi(\xi) = -i\langle \mu, \xi \rangle + \Gamma(-\nu) \int_{\mathbb{S}^{n-1}} ((\lambda(\phi))^\nu - (\lambda(\phi) - i\langle \Sigma\xi, \phi \rangle)^\nu) \Pi'(d\phi),$$

where $\nu \in (0, 2)$, $\mu \in \mathbb{R}^n$ and Σ is a positive-definite matrix. Clearly

$$\psi(\xi) = -i\langle \mu, \xi \rangle + C_1 - C_2(\xi),$$

where

$$C_1 := \int_{\mathbb{S}^{n-1}} (\lambda(\phi))^\nu \Pi'(d\phi)$$

and

$$C_2(\xi) := \int_{\mathbb{S}^{n-1}} (\lambda(\phi) - i\langle \Sigma\xi, \phi \rangle)^\nu \Pi'(d\phi).$$

Let in particular $\Pi'(d\phi) = cd\phi$, where $c > 0$ and $d\phi$ is the Haar measure on \mathbb{S}^{n-1} . Then the problem is to approximate the integral

$$C_2(\xi) := c \int_{\mathbb{S}^{n-1}} (\lambda(\phi) - i \langle \Sigma \xi, \phi \rangle)^\nu d\phi.$$

This problem is computationally difficult. In this section we construct a class of KoBoL processes which are based on the respective one-dimensional blocks. This allows us to simplify the expression of the characteristic exponent.

We start with a one-dimensional version of the Theorem 5,

$$\psi(\xi) = 2^{-1}a\xi^2 - i\gamma\xi - \int_{\mathbb{R}} (\exp(ix\xi) - 1 - ix\xi\chi_{[-1,1]}(x)) \Pi(dx),$$

where $a \geq 0, \gamma \in \mathbb{R}$ and Π is a measure on \mathbb{R} satisfying

$$\Pi(\{0\}) = 0, \int_{\mathbb{R}} \min\{x^2, 1\} \Pi(dx) < \infty.$$

Let $a = \gamma = 0, 0 < \nu < 2, \lambda > 0$,

$$\Pi^+(\nu, \lambda, dx) = x_+^{-\nu-1} \exp(-\lambda x) dx,$$

$$\Pi^-(\nu, \lambda, dx) = x_-^{-\nu-1} \exp(\lambda x) dx,$$

where $x_+ = \max\{x, 0\}, x_- = x_+ - x$ and

$$\Pi(dx) = c_+ \Pi^+(\nu, -\lambda_-, dx) + c_- \Pi^-(\nu, \lambda_+, dx), c_+ > 0, c_- > 0, \lambda_- < 0 < \lambda_+. \quad (6)$$

It is easy to check that

$$\int_{\mathbb{R}} \min\{x^2, 1\} (c_+ \Pi^+(\nu, -\lambda_-, dx) + c_- \Pi^-(\nu, \lambda_+, dx)) < \infty.$$

Hence (6) defines a Lévy measure. Moreover, if $\nu < 1$ then

$$\int_{\mathbb{R}} \min\{|x|, 1\} (c_+ \Pi^+(\nu, -\lambda_-, dx) + c_- \Pi^-(\nu, \lambda_+, dx)) < \infty$$

and the process has a finite variation.

Lemmas 3.1 and 3.2 [2] give a representation of the respective characteristic exponent

$$\begin{aligned} \psi(\xi) &= -i\mu\xi + c_+ \Gamma(-\nu) ((-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu) \\ &\quad + c_- \Gamma(-\nu) (\lambda_+^\nu - (\lambda_+ + i\xi)^\nu), \nu \in (0, 1) \cup (1, 2). \end{aligned} \quad (7)$$

The proof of Lemma 3.2 presented in [2] is incomplete. The next statement gives a complete proof of the representation (7) which is important in our applications.

Theorem 4. *Let $\nu \in (0, 1)$ then in our notations*

$$\psi(\xi) = -i\mu\xi + c_+ \Gamma(-\nu) ((-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu)$$

$$+c_-\Gamma(-\nu)(\lambda_+^\nu - (\lambda_+ + i\xi)^\nu),$$

where μ is a real parameter.

Proof It is sufficient to prove the statement just for the $\Pi^+(\nu, \lambda, dx)$, i.e. to find

$$\begin{aligned} -\psi^+(\xi) &:= \int_{\mathbb{R}} (\exp(ix\xi) - 1 - ix\xi\chi_{[-1,1]}(x)) \Pi^+(dx) \\ &= \int_{\mathbb{R}} (\exp(ix\xi) - 1 - ix\xi\chi_{[-1,1]}(x)) x_+^{-\nu-1} \exp(-\lambda x) dx \\ &= \int_0^\infty (\exp(ix\xi) - 1 - ix\xi\chi_{[-1,1]}(x)) x^{-\nu-1} \exp(-\lambda x) dx \\ &= \int_0^\infty (\exp(ix\xi) - 1) x^{-\nu-1} \exp(-\lambda x) dx \\ &\quad - i\xi \int_0^1 x^{-\nu} \exp(-\lambda x) dx \\ &= \int_0^\infty (\exp(ix\xi) - 1) x^{-\nu-1} \exp(-\lambda x) dx - i\xi B(\nu, \lambda) \\ &:= I_1(\xi, \nu, \lambda) - i\xi B(\nu, \lambda), \end{aligned}$$

where $B(\nu, \lambda) := \int_0^1 x^{-\nu} \exp(-\lambda x) dx$ and

$$\begin{aligned} I_1(\xi, \nu, \lambda) &= -\frac{1}{\nu} \int_0^\infty (\exp(-(\lambda - i\xi)x) - \exp(-\lambda x)) dx x^{-\nu} \\ &= -\frac{1}{\nu} ((\exp(-(\lambda - i\xi)x) - \exp(-\lambda x)) x^{-\nu})|_0^\infty \\ &\quad - \left(-\frac{1}{\nu}\right) \int_0^\infty (-\lambda + i\xi) \exp(-(\lambda - i\xi)x) + \lambda \exp(-\lambda x) dx x^{-\nu} \\ &= -\frac{\lambda - i\xi}{\nu} \int_0^\infty \exp(-(\lambda - i\xi)x) x^{-\nu} dx - \lambda^\nu \Gamma(-\nu) := I_2 - \lambda^\nu \Gamma(-\nu). \end{aligned}$$

Making change of variable $z = (\lambda - i\xi)x$ in I_2 we get

$$I_2 = -\frac{(\lambda - i\xi)^\nu}{\nu} \int_\gamma \exp(-z) z^{-\nu} dz,$$

where γ is the ray $\{z | z = (\lambda - i\xi)x, \lambda > 0, \xi \in \mathbb{R}\}$, λ and ξ are fixed parameters and $x \geq 0$. Assume that $\xi \geq 0$. The case $\xi \leq 0$ can be treated similarly. Consider the contour $\eta := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\gamma_1 := \{z = \rho \exp(i\theta) | 0 \leq \theta \leq \arg(\lambda - i\xi), \lambda > 0, \xi \in \mathbb{R}\},$$

$$\gamma_2 := \{z | \rho \leq z \leq R, z \in \mathbb{R}\}$$

$$\gamma_3 := \{z = R \exp(i\theta) | 0 \leq \theta \leq \arg(\lambda - i\xi), \lambda > 0, \xi \in \mathbb{R}\},$$

$$\gamma_4 := \{z \mid z = (\lambda - i\xi)x, \rho \leq |z| \leq R\}.$$

The function $\exp(-z)z^{-\nu}$ is analytic in the domain bounded by η , hence from the Cauchy's theorem it follows that

$$\int_{\eta} \exp(-z)z^{-\nu} dz = 0$$

and since $\xi \geq 0$ then for some $\delta > 0$ we get $-\pi/2 + \delta \leq \arg(\lambda - i\xi) \leq 0$. Hence

$$\begin{aligned} & \lim_{R \rightarrow \infty} \left| \int_{\gamma_3} \exp(-z)z^{-\nu} dz \right| \\ &= \lim_{R \rightarrow \infty} \left| \int_0^{\arg(\lambda - i\xi)} \exp(-R \exp(i\theta)) R^{-\nu} \exp(-i\nu\theta) Ri \exp(i\theta) d\theta \right| \\ &\leq \frac{\pi}{2} \lim_{R \rightarrow \infty} \exp(-R \cos \delta) \exp(R^{1-\nu}) = 0. \end{aligned}$$

Observe that

$$\begin{aligned} & \lim_{\rho \rightarrow 0} \left| \int_{\gamma_3} \exp(-z)z^{-\nu} dz \right| \\ &\leq \lim_{\rho \rightarrow 0} \left| \int_0^{2\pi} \exp(-\rho \exp(i\theta)) \rho^{-\nu} \exp(-i\nu\theta) \rho i \exp(i\theta) d\theta \right| \\ &\leq 2\pi \lim_{\rho \rightarrow 0} \rho^{-\nu+1} = 0. \end{aligned}$$

Hence

$$\int_{\gamma} \exp(-z)z^{-\nu} dz = \int_{\mathbb{R}_+} \exp(-z)z^{-\nu} dz = \Gamma(-\nu + 1) = -\nu \Gamma(-\nu).$$

Consequently

$$I_2 = -\frac{(\lambda - i\xi)^\nu}{\nu} \int_{\gamma} \exp(-y)y^{-\nu} dy = \Gamma(-\nu)(\lambda - i\xi)^\nu$$

and

$$\psi^+(\xi) = \Gamma(-\nu)(\lambda^\nu - ((\lambda - i\xi)^\nu)) + i\xi B(\nu, \lambda).$$

5 Appendix I: Stochastic processes and density functions

Let $\mathcal{B}(\mathbb{R}^n)$ be the collection of all Borel sets in \mathbb{R}^n (which is the σ -algebra generated by all open sets in \mathbb{R}^n). A mapping $\mathbf{X} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an \mathbb{R}^n -valued random variable if it is $\mathcal{B}(\mathbb{R}^n)$ measurable, i.e. for any $B \in \mathcal{B}(\mathbb{R}^n)$ we have $\{\omega \mid \mathbf{X}(\omega) \in B\} \in \mathcal{B}(\mathbb{R}^n)$. Let $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P)$ be a fixed probability space. A stochastic process $\mathbf{X} = \{\mathbf{X}_t, t \in \mathbb{R}\}$ is a one-parametric family of random

variables on a common probability space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P)$. The trajectory of the process \mathbf{X} is a map

$$\begin{aligned} \mathbb{R}_+ &\longrightarrow \mathbb{R}^n \\ t &\longmapsto \mathbf{X}_t(\omega), \end{aligned}$$

where $\omega \in \Omega$ and $\mathbf{X}_t = (X_{1t}, \dots, X_{nt})$.

$\mathbf{X} = \{\mathbf{X}_t\}_{t \in \mathbb{R}_+}$ is called a Lévy process (process with stationary independent increments) if

1. The random variables $\mathbf{X}_{t_0}, \mathbf{X}_{t_1} - \mathbf{X}_{t_0}, \dots, \mathbf{X}_{t_m} - \mathbf{X}_{t_{m-1}}$, for any $0 \leq t_0 < t_1 < \dots < t_m$ and $m \in \mathbb{N}$ are independent (independent increment property).
2. $\mathbf{X}_0 = \mathbf{0}$ a.s.
3. The distribution of $\mathbf{X}_{t+\tau} - \mathbf{X}_t$ is independent of τ (temporal homogeneity or stationary increments property).
4. It is stochastically continuous, i.e.

$$\lim_{\tau \rightarrow t} P[|\mathbf{X}_\tau - \mathbf{X}_t| > \epsilon] = 0$$

for any $\epsilon > 0$ and $t \geq 0$.

5. There is $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ such that, for any $\omega \in \Omega_0$, $\mathbf{X}_t(\omega)$ is right-continuous on $[0, \infty)$ and has left limits on $(0, \infty)$.

A process satisfying (1 – 4) is called a Lévy process in law. An additive process is a stochastic process which satisfies (1, 2, 4, 5) and an additive process in law satisfies (1, 2, 4).

For an integrable on \mathbb{R}^n function, $f \in L_1(\mathbb{R}^n)$ define its Fourier transform

$$\mathbf{F}f(\mathbf{y}) = \int_{\mathbb{R}^n} \exp(-i \langle \mathbf{x}, \mathbf{y} \rangle) f(\mathbf{x}) d\mathbf{x}$$

and its formal inverse

$$(\mathbf{F}^{-1}f)(\mathbf{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(i \langle \mathbf{x}, \mathbf{y} \rangle) f(\mathbf{y}) d\mathbf{y}.$$

Let $\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{k=1}^n u_k v_k$ be the canonic scalar product on \mathbb{R}^n , $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$. The characteristic function of the distribution of \mathbf{X}_t of any Lévy process can be represented in the form

$$\begin{aligned} \mathbb{E}[\exp(\langle i\mathbf{x}, \mathbf{X}_t \rangle)] &= e^{-t\psi(\mathbf{x})} \\ &= (2\pi)^n \mathbf{F}^{-1}p_t(\mathbf{x}), \end{aligned}$$

where $p_t(\mathbf{x})$ is the density function of \mathbf{X}_t , $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ and the function $\psi(\mathbf{x})$ is uniquely determined. This function is called the characteristic exponent. Vice

versa, a Lévy process $\mathbf{X} = \{\mathbf{X}_t\}_{t \in \mathbb{R}_+}$ is determined uniquely by its characteristic exponent $\psi(\mathbf{x})$. In particular, density function p_t can be expressed as

$$p_t(\cdot) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(-i\langle \cdot, \mathbf{x} \rangle - t\psi(\mathbf{x})) d\mathbf{x} = (2\pi)^{-n} \mathbf{F}(\exp(-t\psi(\mathbf{x}))) (\cdot).$$

The key role in our analysis plays the following classical result known as the Lévy-Khintchine formula which gives a representation of characteristic functions of all infinitely divisible distributions.

Theorem 5. *Let $\mathbf{X} = \{\mathbf{X}_t\}_{t \in \mathbb{R}_+}$ be a Lévy process on \mathbb{R}^n . Then its characteristic exponent admits the representation*

$$\psi(\mathbf{y}) = 2^{-1} \langle A\mathbf{y}, \mathbf{y} \rangle - i\langle \mathbf{b}, \mathbf{y} \rangle - \int_{\mathbb{R}^n} \left(e^{i\langle \mathbf{y}, \mathbf{x} \rangle} - 1 - i\langle \mathbf{y}, \mathbf{x} \rangle \chi_D(\mathbf{x}) \right) \Pi(d\mathbf{x}), \quad (8)$$

where $\chi_D(\mathbf{x})$ is the characteristic function of $D := \{\mathbf{x} \in \mathbb{R}^n, |\mathbf{x}| \leq 1\}$, A is a symmetric nonnegative-definite $n \times n$ matrix, $\mathbf{b} \in \mathbb{R}^n$ and $\Pi(d\mathbf{x})$ is a measure on \mathbb{R}^n such that

$$\int_{\mathbb{R}^n} \min\{1, \langle \mathbf{x}, \mathbf{x} \rangle\} \Pi(d\mathbf{x}) < \infty, \quad \Pi(\{\mathbf{0}\}) = 0.$$

Hence $\hat{\mu}(\mathbf{y}) = e^{\psi(\mathbf{y})}$.

The density of Π is known as the Lévy density and A is the covariance matrix. In particular, if $A = 0$ (i.e. $A = (a_{j,k})_{1 \leq j,k \leq n}$, $a_{j,k} = 0$) then the Lévy process is a pure non-Gaussian process and if $\Pi = 0$ the process is Gaussian. See [9]-[11], [22], [25] for more information.

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